J. Math. Anal. Appl. 385 (2012) 672–676

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lower bounds for blow-up in a model of chemotaxis

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ARTICLE INFO

Article history:
Received 24 November 2010
Available online 6 July 2011
Submitted by P. Sacks

Keywords:
Lower bounds
Blow-up time
Parabolic–parabolic system
Chemotaxis

ABSTRACT

Many special cases of the classical Keller–Segel system for modeling chemotaxis have been investigated in the literature, and typically the solution of the governing equations will blow up at some finite time. However, the question of establishing lower bounds for this blow-up time has been largely ignored. This paper derives such a lower bound in a parabolic–parabolic model in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

1. Introduction

A basic system of equations modeling chemotaxis was established in 1970 by Keller and Segel [7] (see also [8,9]), and since that time numerous papers on chemotaxis have appeared in both the mathematical and the biological literatures. These papers have dealt primarily with the qualitative properties of the solutions to various special cases of the Keller–Segel system. Much of the work prior to 2003 is referenced in the papers of Horstmann [4,5] (see also the book of Straughan [14]). Many of the more recent investigations are referred to in a paper of Hillen and Painter [3]. We mention also the recent work of Corrias and Pertame [2] and of Payne and Straughan [13]. Recent papers treating the problem in two dimensions have been published by Blanchet et al. [1] and Kozono and Sugiyama [10].

To our knowledge none of the above mentioned papers have dealt with the question of determining lower bounds for the time of blow-up in those cases in which the solution blows up at some finite time. A recent paper of Payne and Song [12] established such a lower bound for the special parabolic–elliptic system studied by Jäger and Luckhaus [6]. In the present paper the authors derive a lower bound for the blow-up time for the more general parabolic–parabolic system.

The Keller–Segel equations involve a population (concentration u) and a chemotactic agent (concentration v), and we consider here the special case of their system in which the non-negative functions u and v satisfy equations

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - k_1 (u v_{,i})_{,i} \\ \frac{\partial v}{\partial t} &= k_2 \Delta v - k_3 v + k_4 u \end{aligned}\right\} \quad \text{in } \Omega \times (0, t^*), \tag{1.1}$$

where t^* is the time of blow-up, Ω is a bounded convex region in either $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, Δ is the Laplace operator, and k_i $(i = 1, 2, 3, 4)$ are positive constants. Here and throughout a comma is used to denote differentiation and the convention of summing over repeated subscripts from 1 to 3 is adopted.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.086

Associated with the system are the boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \Omega \times \left(0, t^*\right), \tag{1.2}$$

where $\partial/\partial n$ stands for the normal derivative on $\partial\Omega$. In addition, the non-negative functions u and v satisfy the initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \geqslant 0, \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}) \geqslant 0. \tag{1.3}$$

Here u is a continuous function and v is a C^2 function in Ω with v satisfying appropriate compatibility on $\partial\Omega$.

2. Blow-up time in $\mathbb{R}^3$

To derive a lower bound for t^* we introduce the function $\phi(t)$ given by

$$\phi(t) = \alpha \int_\Omega u^2\,dx + \int_\Omega (\Delta v)^2\,dx \tag{2.1}$$

for some positive constant α to be determined. Then assuming the solution of (1.1)–(1.3) blows up in ϕ measure at time t^*, we establish a lower bound for the blow-up time.

Differentiating we have

$$\begin{aligned}
\frac{d\phi}{dt} &= 2\alpha \int_\Omega u\left[\Delta u - k_1 (u v_{,i})_{,i}\right] dx + 2\int_\Omega \Delta v \Delta v_{,t}\,dx \\
&= -2\alpha \int_\Omega |\nabla u|^2\,dx - \alpha k_1 \int_\Omega u^2 \Delta v\,dx - 2\int_\Omega \Delta v_{,i} v_{,it}\,dx \\
&= -2\alpha \int_\Omega |\nabla u|^2\,dx - \alpha k_1 \int_\Omega u^2 \Delta v\,dx - 2k_2 \int_\Omega \Delta v_{,i} \Delta v_{,i}\,dx - 2k_3 \int_\Omega (\Delta v)^2\,dx - 2k_4 \int_\Omega \Delta v_{,i} u_{,i}\,dx \\
&\leqslant -2\alpha \int_\Omega |\nabla u|^2\,dx - \alpha k_1 \int_\Omega u^2 \Delta v\,dx - 2k_2 \int_\Omega \Delta v_{,i} \Delta v_{,i}\,dx - 2k_3 \int_\Omega (\Delta v)^2\,dx + k_4 \epsilon_1 \int_\Omega \Delta v_{,i} \Delta v_{,i}\,dx \\
&\quad + \frac{k_4}{\epsilon_1} \int_\Omega (\nabla u)^2\,dx,
\end{aligned} \tag{2.2}$$

where ∇ is the gradient operator and the arithmetic–geometric mean inequality with a positive, as yet unspecified, weight ϵ_1 is used.

We now focus our attention on the second term on the right in (2.2) and use Hölder's inequality to bound

$$\int_\Omega u^2 \Delta v\,dx \leqslant \left[\int_\Omega u^3\,dx\right]^{2/3} \left[\int_\Omega |\Delta v|^3\,dx\right]^{1/3}. \tag{2.3}$$

We now use the fundamental inequality

$$a^p b^{1-p} \leqslant pa + (1-p)b \tag{2.4}$$

with an undetermined positive weight factor ϵ_2

$$\left[\int_\Omega u^3\,dx\right]^{2/3} \left[\int_\Omega |\Delta v|^3\,dx\right]^{1/3} \leqslant \frac{2\epsilon_2}{3} \int_\Omega u^3\,dx + \frac{1}{3\epsilon_2^2} \int_\Omega |\Delta v|^3\,dx. \tag{2.5}$$

Substituting (2.5) and (2.3) into (2.2), we have

$$\begin{aligned}
\frac{d\phi}{dt} &\leqslant -2\alpha \int_\Omega |\nabla u|^2\,dx + \alpha k_1 \left\{\frac{2\epsilon_2}{3} \int_\Omega u^3\,dx + \frac{1}{3\epsilon_2^2} \int_\Omega |\Delta v|^3\,dx\right\} \\
&\quad - (2k_2 - k_4\epsilon_1) \int_\Omega \Delta v_{,i} \Delta v_{,i}\,dx - 2k_3 \int_\Omega (\Delta v)^2\,dx + \frac{k_4}{\epsilon_1} \int_\Omega (\nabla u)^2\,dx.
\end{aligned} \tag{2.6}$$

To bound the second term on the right in (2.6) in terms of ϕ, $\int_\Omega |\nabla u|^2\,dx$, and $\int_\Omega \Delta v_{,i}\Delta v_{,i}\,dx$, we make use of an inequality (2.16) in Payne and Schaefer [11], i.e.

$$\begin{aligned}\int_\Omega u^3\,dx &\leqslant \left[m_1\int_\Omega u^2\,dx + m_2\left(\int_\Omega u^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}\right]^{3/2}\\ &\leqslant 2^{1/2}\left[m_1^{3/2}\left(\int_\Omega u^2\,dx\right)^{3/2} + m_2^{3/2}\left\{\left(\int_\Omega u^2\,dx\right)^3\right\}^{1/4}\left(\int_\Omega |\nabla u|^2\,dx\right)^{3/4}\right]\\ &\leqslant 2^{1/2}\left[m_1^{3/2}\left(\int_\Omega u^2\,dx\right)^{3/2} + m_2^{3/2}\left\{\frac{1}{4\epsilon_3^3}\left(\int_\Omega u^2\,dx\right)^3 + \frac{3\epsilon_3}{4}\int_\Omega |\nabla u|^2\,dx\right\}\right],\end{aligned} \tag{2.7}$$

where

$$m_1 = \frac{1}{2\,3^{1/8}p_0}, \qquad m_2 = \frac{1}{3^{9/8}}\left(\frac{d}{p_0}+1\right), \tag{2.8}$$

and for some origin inside Ω

$$p_0 = \min_{\partial\Omega} x_i n_i > 0, \qquad d^2 = \max_{\overline{\Omega}} x_i x_i, \tag{2.9}$$

n_i being the ith component of the unit normal vector directed outward on $\partial\Omega$. In (2.7) we have used the fact that for positive a and b

$$(a+b)^{3/2} \leqslant 2^{1/2}\big(a^{3/2}+b^{3/2}\big). \tag{2.10}$$

Similarly, since Payne and Schaefer's result clearly holds for absolute values

$$\int_\Omega |\Delta v|^3\,dx \leqslant 2^{1/2}\left[m_1^{3/2}\left(\int_\Omega (\Delta v)^2\,dx\right)^{3/2} + m_2^{3/2}\left\{\frac{1}{4\epsilon_4^3}\left(\int_\Omega (\Delta v)^2\,dx\right)^3 + \frac{3\epsilon_4}{4}\int_\Omega \Delta v_{,i}\Delta v_{,i}\,dx\right\}\right]. \tag{2.11}$$

On substituting (2.7) and (2.11) into (2.6), we obtain the differential inequality

$$\begin{aligned}\frac{d\phi}{dt} \leqslant{}& \left[-2\alpha + \frac{1}{\sqrt{2}}m_2^{3/2}k_1\alpha\epsilon_2\epsilon_3 + \frac{k_4}{\epsilon_1}\right]\int_\Omega |\nabla u|^2\,dx + \left[-2k_2 + k_4\epsilon_1 + \sqrt{2}k_1\alpha m_2^{3/2}\frac{\epsilon_4}{4\epsilon_2^2}\right]\int_\Omega \Delta v_{,i}\Delta v_{,i}\,dx\\ &+ \frac{2}{3}k_1\alpha\epsilon_2\left[\sqrt{2}m_1^{3/2}\left(\int_\Omega u^2\,dx\right)^{3/2} + \frac{\sqrt{2}m_2^{3/2}}{4\epsilon_3^3}\left(\int_\Omega u^2\,dx\right)^3\right] + \frac{\sqrt{2}k_1\alpha m_1^{3/2}}{3\epsilon_2^2}\left[\int (\Delta v)^2\,dx\right]^{3/2}\\ &+ \frac{\sqrt{2}k_1\alpha m_2^{3/2}}{12\epsilon_2^2\epsilon_4^3}\left[\int (\Delta v)^2\,dx\right]^3 - 2k_3\int (\Delta v)^2\,dx.\end{aligned} \tag{2.12}$$

We drop the last term on the right side and choose α and the ϵ_i $(i=1,2,3,4)$ such that

$$\begin{aligned}&-2\alpha + \frac{1}{\sqrt{2}}m_2^{3/2}k_1\alpha\epsilon_2\epsilon_3 + \frac{k_4}{\epsilon_1} \leqslant 0,\\ &-2k_2 + k_4\epsilon_1 + \sqrt{2}k_1\alpha m_2^{3/2}\frac{\epsilon_4}{4\epsilon_2^2} \leqslant 0,\end{aligned} \tag{2.13}$$

and arrive at

$$\frac{d\phi}{dt} \leqslant A\phi^{3/2} + B\phi^3 \tag{2.14}$$

where A and B are computable constants depending on the choices made for α and the ϵ_i. We have made use of the fact that for $\gamma > 1$ and a and b non-negative

$$a^\gamma + b^\gamma \leqslant (a+b)^\gamma. \tag{2.15}$$

A possible choice for α and the ϵ_i is

$$\epsilon_1 = \frac{k_2}{k_4}, \qquad \epsilon_2 = 1, \qquad \epsilon_3 = \frac{\sqrt{2}}{k_1 m_2^{3/2}}, \qquad \epsilon_4 = \frac{2\sqrt{2}k_2^2}{k_1 k_4^2 m_2^{3/2}}, \qquad \alpha = \frac{k_4^2}{k_2}. \tag{2.16}$$

An integration of (2.14) leads to

$$t \geqslant \int_{\phi(0)}^{\phi(t)} \frac{d\eta}{A\eta^{3/2} + B\eta^3}, \tag{2.17}$$

and if $\phi(t)$ blows up at time t^* then

$$t^* \geqslant \int_{\phi(0)}^{\infty} \frac{d\eta}{A\eta^{3/2} + B\eta^3}, \tag{2.18}$$

where

$$\phi(0) = \alpha \int_{\Omega} u_0^2 \, dx + \int (\Delta v_0)^2 \, dx.$$

We have established the following theorem.

Theorem 1. *Let (u, v) be the solution of* (1.1)–(1.3) *in a convex region Ω in $\mathbb{R}^3$ with smooth boundary and compatible data, and suppose the solution blows up in ϕ measure at time t^*, then t^* satisfies the lower bound* (2.18)*.*

The convexity in $\mathbb{R}^3$ was used in the derivation of (2.7). However, the derivation of the theorem requires only that Ω be star shaped and convex separately in two orthogonal directions, so our result holds for these more general regions.

We remark that the integrals in (2.18) can either be evaluated or easily bounded from below. It is also worth noting that if it is not known whether the solution blows up or not, our bound will assure us of a safe time period in which blow-up cannot occur.

3. Blow-up time in $\mathbb{R}^2$

Since several papers have dealt with the two-dimensional version of this problem we here indicate the changes that must be made from our derivation in $\mathbb{R}^3$. We use the same form for ϕ, with integrals now defined on a two-dimensional domain D with boundary ∂D. The arguments through (2.2) are the same as before. However, (2.3) and (2.11) will be changed. In the derivation of (2.16) in [11], the authors make use of an inequality (see the inequality following (2.10) in [11]), which in our notation is

$$\left(\int_D u^4 \, dA \right)^{\frac{1}{2}} \leqslant \left(\frac{1}{2} \oint_{\partial D} u^2 |n_x| \, ds + \int_D u|u_{,x}| \, dA \right)^{1/2} \times \left(\frac{1}{2} \oint_{\partial D} u^2 |n_y| \, ds + \int_D u|u_{,y}| \, dA \right)^{1/2}. \tag{3.1}$$

Making use of the arithmetic–geometric mean inequality and the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\left(\int_D u^4 \, dA \right)^{1/2} &\leqslant \frac{1}{4} \left(\oint_{\partial D} u^2 |n_x| \, ds + \oint_{\partial D} u^2 |n_y| \, ds \right) + \frac{1}{2} \left(\int_D u|u_{,x}| \, dA + \int_D u|u_{,y}| \, dA \right) \\
&\leqslant \frac{1}{4} \left\{ \left(\oint_{\partial D} u^2 \, ds \oint_{\partial D} u^2 |n_x|^2 \, ds \right)^{1/2} + \left(\oint_{\partial D} u^2 \, ds \oint_{\partial D} u^2 |n_y|^2 \, ds \right)^{1/2} \right\} \\
&\quad + \frac{1}{2} \left\{ \left(\int_D u^2 \, dA \int_D u_{,x}^2 \, dA \right)^{1/2} + \left(\int_D u^2 \, dA \int_D u_{,y}^2 \, dA \right)^{1/2} \right\} \\
&\leqslant \frac{\sqrt{2}}{4} \oint_{\partial D} u^2 \, ds + \frac{\sqrt{2}}{2} \left(\int_D u^2 \, dA \right)^{1/2} \left(\int_D |\nabla u|^2 \, dA \right)^{1/2}.
\end{aligned} \tag{3.2}$$

In the last step we have used the fact that for non-negative a and b

$$a^{1/2} + b^{1/2} \leqslant \sqrt{2}(a+b)^{1/2}. \tag{3.3}$$

Again, since D is assumed to be convex, it follows that

$$\oint_{\partial D} u^2 \, ds \leqslant \frac{2}{p_0} \int_D u^2 \, dA + \frac{2d}{p_0} \left(\int_D u^2 \, dA \int_D |\nabla u|^2 \, dA \right)^{1/2}, \tag{3.4}$$

where as before

$$p_0 = \min_{\partial\Omega} x_\beta n_\beta > 0, \qquad d^2 = \max_{\overline{\Omega}} x_\beta x_\beta, \quad \beta = 1, 2. \tag{3.5}$$

Inserting (3.4) back into (3.2) and making use of the inequality

$$\int_D u^3\, dA \leqslant \left(\int_D u^2\, dA \int_D u^4\, dA \right)^{1/2} \tag{3.6}$$

leads to the bounds

$$\int_D u^3\, dA \leqslant \frac{\sqrt{2}}{2p_0}\left(\int_D u^2\, dA \right)^{3/2} + \frac{\sqrt{2}}{2}\left(1 + \frac{d}{p_0}\right) \int_D u^2\, dA \left(\int_D |\nabla u|^2\, dA \right)^{1/2} \tag{3.7}$$

and

$$\int_D |\Delta v|^3\, dA \leqslant \frac{\sqrt{2}}{2p_0}\left(\int_D |\Delta v|^2\, dA \right)^{3/2} + \frac{\sqrt{2}}{2}\left(1 + \frac{d}{p_0}\right) \int_D |\Delta v|^2\, dA \left(\int_D \Delta v_{,i} \Delta v_{,i}\, dA \right)^{1/2}. \tag{3.8}$$

Following the arguments used for $\mathbb{R}^3$ we arrive at

$$t^* \geqslant \int_{\phi(0)}^{\infty} \frac{d\eta}{A_1 \eta^{3/2} + B_1 \eta^2}, \tag{3.9}$$

where again the values of A_1 and B_1 are easily obtained, and the integral is easily evaluated. We have established the following theorem.

Theorem 2. *Let (u, v) be the solution of* (1.1)–(1.3) *in a convex region D in $\mathbb{R}^2$ with smooth boundary and compatible data. Then if the solution blows up in ϕ measure at time t^*, it follows that t^* satisfies the lower bound* (3.9).

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF 2010-0012215).

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